4.3: Monotonic Fcns and First Derivative Test

Corollary 3 (to MVT):

If $f(x)$ is increasing then $f'(x) > 0$

If $f(x)$ is decreasing then $f'(x) < 0$.

Thus, if x is a relative extrema and $f'(x)$ exists then $f'(x) = 0$.

Ex(1): Find the critical points of $f(x) = x^3 - 12x - 5$.

Then $f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x-2)(x+2)$.

$f'(x) = 0$ when $x = \pm 2$. These are critical pts.

First Derivative Test for local extrema:

Suppose c is a critical pt of f and f is diff. in some interval containing c.

1. If f' changes from negative to positive at c, then f has a local ~~maximum~~ minimum at c.
2. If f' changes from positive to negative at c, then f has a local max at c.
3. If the sign of f' does not change, then f has no local extremum at c.

Ex(2): Find local extrema of $f(x) = x^{1/3}(x-4)$.

Well, $f(x) = x^{4/3} - 4x^{1/3}$ so $f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x-1)$.

So $f'(x) = 0$ when $x = 1$ and undefined at $x = 0$

Furthermore, $f'(1/2) < 0$ and $f'(2) > 0$ and $f'(-1) < 0$

So 1 is a local ~~maximum~~ minimum of f.

Ex(3): Find local extrema of $f(x) = (x^2-3)e^x$.

Then $f'(x) = (x^2-3)e^x + 2xe^x = e^x(x^2+2x-3) = e^x(x+3)(x-1)$.

So critical pts at $x = -3, 1$.

Also, $f'(-4) > 0$, $f'(0) < 0$, $f'(2) > 0$

So f has local max at $x = -3$

and local min at $x = 1$.

4.4: Concavity and Curve Sketching

Concave Up

⇓

f' is increasing

⇓

$f''>0$

Concave Down

⇓

f' is decreasing

⇓

$f''<0$

Def^n: f is concave up if f' is increasing

f is concave down if f' is decreasing

Second Derivative Test:

(a) $f''>0$ then f is concave up

$f''<0$ then f is concave down.

(b) If c is a critical pt of f,

$f''(c)>0$ implies c is a local min.

$f''(c)<0$ implies c is a local max.

Def^n: If c is a point where concavity changes then c is called an inflection point. Then $f''(c)=0$ or undefined.

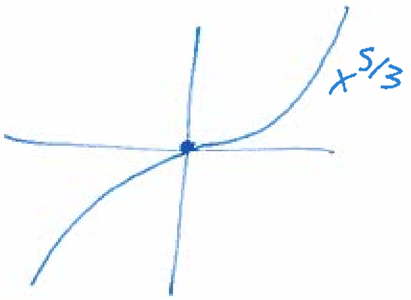
Ex(1): Find all critical and inflection points. Determine the points which are local extrema. Sketch the graph of f.

(a) $f(x)=x^{5/3}$. $\Rightarrow f'(x)=\frac{5}{3}x^{2/3} \Rightarrow f''(x)=\frac{10}{9}x^{-1/3}=\frac{10}{9\sqrt[3]{x}}$.

Critical pt: $f'(x)=0 \Rightarrow x=0$. $f'(-1)>0$, $f'(1)>0$ so no local extrema

Inflection pt: $f''(x)$ is undefined at 0.

$f''(-1)<0$ and $f''(1)>0$ so 0 is an inflection pt.



(b) $f(x)=x^4 \Rightarrow f'(x)=4x^3 \Rightarrow f''(x)=12x^2$

Crit. pt: $f'(x)=0 \Rightarrow x=0$. ~~$f'(-1)<0, f'(1)>0$ so~~

Inflection pt: $f''(x)=0 \Rightarrow x=0$. $f''(-1)>0$, $f''(1)>0$ so no sign change and thus no inflection pt.

Also $f''(0)=0$ so 2nd der. test fails.
We must use first der. test.

$f'(-1)<0$, $f'(1)>0$ so 0 is a local min of f.